UCRL- 83504
PREPRINT

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This paper was prepared for submittal to
Physics of Fluids

October 23, 1979

TEARING-MODE STABILITY ANALYSIS OF A CYLINDRICAL PLASMA*

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ABSTRACT

An investigation of tearing mode stability for a cylindrical plasma finds that stability improves greatly as flux is excluded from the equilibrium and provided that the inner plasma surface is close to the axis.

Reversed field θ-pinch experiments exhibit long lifetimes,[1-3] and the question of stability to the tearing mode frequently arises. In this note, we use neighboring equilibrium arguments originally developed by Pfirsch[4] to show that tearing-mode stability is strongly enhanced if two conditions are fulfilled: the first that the flux tend to be excluded in the bulk of the plasma, and the second that the plasma edge lie close to the axis. Such conditions tend to be fulfilled in reversed-field θ-pinch experiments and may be responsible for lack of observations of tearing instability. This beneficial effect can also arise in steady-state plasmas with a significant bootstrap effect, where the flux is low throughout the body of the plasma.

*Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore Laboratory under contract number W-7405-ENG-48.

First, let us consider a class of MHD cylindrical equilibria for the flux ψ that satisfy the Grad-Shafranov equation,

$$r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial \psi}{\partial r} + r^2 \frac{\partial p}{\partial \psi} = 0 \quad , \tag{1}$$

where p is the pressure, and the magnetic field $B_z = \frac{1}{r} \frac{\partial \psi}{\partial r}$. We choose

$$\frac{\partial p}{\partial \psi} = - \alpha^2 \psi \, \Theta(\psi_e - \psi) \quad , \tag{2}$$

where ψ_e is the flux at the plasma edge.

The solution to Eq. (2) is

$$\psi_e = \frac{B_e}{\alpha} \coth [\alpha(r_0^2 - r_e^2)/2] \quad ,$$

$$\psi = - \frac{B_e}{2} (r^2 - r_e^2) + \frac{B_e}{\alpha} \coth [\alpha(r_0^2 - r_e^2)/2] \qquad r^2 < r_e^2 \tag{3a}$$

$$B = -B_e$$

$$\psi = \frac{B_e \cosh [\alpha(r^2 - r_0^2)/2]}{\sinh [\alpha(r_0^2 - r_e^2)/2]}$$

$$r_e^2 < r^2 < 2r_0^2 - r_e^2 \tag{3b}$$

$$B = \frac{B_e \sinh [\alpha(r^2 - r_0^2)/2]}{\sinh [\alpha(r_0^2 - r_e^2)/2]}$$

$$\psi = \frac{B_e}{\alpha} \coth [\alpha(r_0^2 - r_e^2)/2] + \frac{B_e}{2} (r^2 - 2r_0^2 + r_e^2) \quad ,$$

$$r_{ed}^2 \equiv 2r_0^2 - r_e^2 < r^2 < r_w^2 \tag{3c}$$

$$B = B_e \quad .$$

Here r_0 is the position of the field null, and the inner and outer boundaries of the plasma are $r^2 = r_e^2$, $2r_0^2 - r_e^2$, respectively. We assume the presence of a wall ($r^2 = r_w^2$) that traps the flux of the plasma.

To apply the neighboring equilibrium method, we need to consider perturbations of the form $\delta\psi = \delta\psi_k(r) \exp(ikz)$ that satisfy the equation

$$r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} \delta\psi - k^2 \delta\psi = -r^2 \frac{\partial^2 p}{\partial \psi^2} \delta\psi \quad , \tag{4}$$

with the boundary condition $\delta\psi(r^2 = 0) = 0$. If we can show that $\delta\psi(r^2)$ has no nulls between $0 \leq r^2 \leq r_w^2$, then the system is stable to tearing. (This is the same technique used by Marx.[5])

We now have, from Eq. (2),

$$\begin{aligned} \frac{\partial^2 p}{\partial \psi^2} &= - \alpha^2 \theta(\psi_e - \psi_1) + \alpha^2 \delta(\psi - \psi_e)\, \psi_e \\ &= - \alpha^2 \theta(r^2 - r_e^2)\, \theta(2r_0^2 - r_e^2 - r^2) \\ &\quad + \frac{\alpha^2 \psi_e}{r B_e} \left\{ \delta(r - r_e) + \delta[r - (2r_0^2 - r_e^2)^{1/2}] \right\} \end{aligned} \tag{5}$$

Hence the equation for $\delta\psi$ becomes,

$$\begin{aligned} &\frac{1}{r} \frac{d}{dr} \left(\frac{1}{r} \frac{d\delta\psi}{dr}\right) - \alpha^2 \delta\psi\, \theta(r^2 - r_0^2)\, \theta(2r_0^2 - r_e^2 - r^2) - k^2 \delta\psi / r^2 \\ &\qquad = - \frac{\alpha^2 \delta\psi}{r B_e} \psi_e \left\{ \delta(r - r_e) + \delta[r - (2r_0^2 - r_e^2)^{1/2}] \right\} \quad . \end{aligned} \tag{6}$$

The easiest case to analyze (and the most difficult to stabilize) is $k = 0$, and we now limit ourselves to this case.

The solution to Eq. (6) is

$$\delta\psi^- = r^2/2 \quad , \qquad r^2 < r_e^2 \tag{7a}$$

$$\delta\psi^0 = r_e^2/2 + \frac{a}{\alpha} \sinh [\alpha(r^2 - r_e^2)/2] \quad , \qquad r_e^2 < r^2 < r_{ed}^2 \tag{7b}$$

$$\delta\psi^+ = \frac{r_e^2}{2} + \frac{a}{\alpha} \sinh [\alpha(r_0^2 - r_e^2)] + \frac{b}{2} (r^2 - r_{ed}^2) \quad , \qquad r_{ed}^2 < r^2 < r_w^2 \quad . \tag{7c}$$

From Eq. (6) we also have the jump conditions

$$\left. \frac{d\delta\psi^+}{dr} - \frac{d\delta\psi^0}{dr} \right|_{r=r_{ed}} = - \alpha \, r_{ed} \coth [\alpha(r_0^2 - r_e^2)/2] \, \delta\psi^+(r_{ed})$$

$$\left. \frac{d\delta\psi^0}{dr} - \frac{d\delta\psi^-}{dr} \right|_{r=r_e} = -\alpha r_e \coth [\alpha(r_0^2 - r_e^2)/2] \, \delta\psi^-(r_e) \quad , \tag{8}$$

which leads to

$$a = 1 - \frac{\alpha r_e^2}{2} \coth \left[\frac{\alpha(r_0^2 - r_e^2)}{2} \right] \quad , \quad b = -1 \quad . \tag{9}$$

The stability of the system demands that $\delta\psi^+(r_w) \geq 0$ with no nodes for $r < r_w$. Solving for r_w in Eq. (7c), by using Eq. (9), leads to

$$\alpha r_w^2 \leq 2\alpha r_0^2 + 2 \sinh [\alpha(r_0^2 - r_e^2)] \left\{ 1 - \frac{\alpha r_e^2}{2} \coth \left[\frac{\alpha(r_0^2 - r_e^2)}{2} \right] \right\} \quad . \tag{10}$$

The most dramatic aspect of this result is the improvement of the stability criteria for large α if $\alpha r_e^2 < 2$. If we first consider $r_e = 0$, we have

$$r_w^2 \leq 2r_0^2 + \frac{2}{\alpha} \sinh (\alpha r_0^2) \quad , \tag{11}$$

so that

$$\frac{r_w^2}{2r_0^2} \longrightarrow \begin{cases} 2 \quad , & \alpha r_0^2 \to 0 \\ \frac{1}{\alpha r_0^2} \exp\left(\alpha r_0^2\right) \quad , & \alpha r_0^2 \gg 1 \quad , \end{cases} \tag{12}$$

which shows that the wall can be exponentially far as α becomes large. Physically, α increasing corresponds to increasing the flux exclusion in the plasma. The flux contained in the plasma is $\psi(r_e) - \psi(r_0)$, which from Eq. (3b) gives,

$$\psi(r_e) - \psi(r_0) \xrightarrow[\alpha r_0^2 \to 0]{} \frac{B_e}{4}\left(r_0^2 - r_e^2\right)$$

$$\xrightarrow[\alpha r_0^2 \gg 1]{} \frac{B_e}{\alpha}$$

The stabilizing property of flux exclusion disappears if the plasma edge is away from the axis, i.e., $\alpha r_e^2/2 \gtrsim 1$. Assuming $\alpha r_0^2 \gg 1$, αr_e^2, Eq. (10) is approximately

$$r_w^2 \leq r_{ed}^2 + r_e^2 - \frac{2}{\alpha}\left(\frac{\alpha r_e^2}{2} - 1\right) \sinh\left(\alpha r_0^2\right) \quad . \tag{13}$$

Then a wall outside the plasma can stabilize only when

$$\frac{\alpha r_e^2}{2} < \frac{1}{1 - \frac{1}{\sinh \alpha r_0^2}} \approx 1 \quad . \tag{14}$$

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